# Arbitrary Order Perturbation Theory for Time-Discrete Vlasov Systems with Drift Maps and Poisson Type Collective Kick Maps

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The well established model[1–3] for studying the micro-bunching instability driven by longitudinal space charge in ultra-relativistic bunches in FEL-like beamlines can be identified as a time-discrete Vlasov model with general drift maps and Poisson type collective kick maps. This model can in principle be solved exactly using the method of characteristics (Perron-Frobenius operator method). Here we describe a higher order perturbative approach based on the Frechet derivative of the Perron-Frobenius operator, and show that is in principle suited to compute analytic approximations to the micro-bunching gain functions.

## 1. Introduction

In linac driven high-gain FELs the generation of bunches with peak current sufficient to support the necessary FEL-gain is typically achieved by creation of a moderately short bunch at the electron source followed by several (typically 1 to 3) stages of bunch-(length) compression interleaved with acceleration. Thereby the charge density of the bunches can be kept acceptable at the lower energies so that emittance blow up through space charge forces is manageable. Multistage compression is often considered beneficial because it minimizes the heating of the bunch through incoherent synchrotron radiation at higher energies as well as the absolute correlated energy spread due to the $E$-chirp needed in the compression process.

However, as we will see, the beamlines, needed to (length)-compress† the bunches, implement several mechanisms to amplify initially small inhomogeneities of the phase-space densities (*PSD*s) into performance degrading substructures (micro-bunches). We will refer to this effect as *micro-bunching amplification*, or short as *micro-bunching*. In this paper we will restrict our study to longitudinal 2-dimensional phase space. Since electrons in a linac quickly reach the ultra-relativistic limit, the compression is often realized by firstly stamping the bunch with a negative (≡ "nose-down") correlation between the particle energy and the longitudinal position inside the bunch through deliberately *chirped* RF accelerating modules and secondly transporting the bunch through a magnetic chicane with pos-

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†We mention here (for the last time) that "bunch-compression" is in fact a sheared rotation in longitudinal phase space which is fully symplectic and thus measure preserving.

itive ($\equiv$ "higher energies pass quicker") longitudinal dispersion (often referred to as "$R_{56}$"). The above procedure typically implies a beamline layout which consists of "long" linac sections, and comparatively "short" chicanes. During passage through the long linac sections various kinds of longitudinal collective intra-bunch effects act on the bunch PSD, thereby potentially modifying the energy distribution of the bunch in a way that depends on the charge distribution. The chicanes supply the longitudinal dispersion and thus the mechanism for modifying the charge density depending on the energy distribution. The most prominent collective effects in the field of micro-bunching are longitudinal space charge (*LSC*) which acts along the complete beamline and coherent synchrotron radiation (*CSR*) which acts solely in the chicanes. We note, however, that any longitudinal collective effect whose spectrum covers the range relevant for intra-bunch substructures can in principle drive the conversion of a density modulation into an energy modulation, i.e. the first step of the micro-bunching process. CSR is essentially an effect in 4-dim phase space (longitudinal $\otimes$ radial) although an approximation via a an impedance in plain longitudinal phase space exists and is well known[4,5].

Here, we will *neglect* CSR and all synchrotron radiation effects, and *restrict* the LSC effects to the long linac section, because by doing so we obtain a model that is **piece-wise exactly integrable**. The dynamical system under study becomes reducible to algebraic relations and integrals and thus allows for explicitly writing down the phase flow as composition of explicitly computable maps, namely the flows over the domains of integrability!

In addition to the above arguments it sometimes appears to helpful to divide a real-live phenomenon (micro-bunching) into distinct self-contained aspects (the gain models with various distinct drivers) and study their dynamics one at a time.

## 2. The Base Model

Our model is purely longitudinal (2-dim phase space) with self-consistent LSC but without any synchrotron radiation effects, i.p. without CSR. We use Cartesian conjugate coordinates $q := -c\tau$ and $p := P_z - P_0 \approx E - E_0$. $\vec{z} := (q,p)^{\mathrm{T}}$, where $\tau$ is the lag of the trajectory w.r.t. the reference trajectory (i.e. the head of the bunch has positive $q$), $c$ describes the clumsiness of the mksA (SI) system of units when it comes to describe atomistic processes ($c = 1$ in atomistic units), $P_z$ and $E$ are the trajectories longitudinal momentum and kinetic energy respectively and $P_0$, $E_0$ are the reference momentum and reference kinetic energy. We employ the ultra-relativistic limit, $\beta \to 1$, $\beta\gamma \to \gamma$, $\gamma \gg 1$ i.p. we neglect the longitudinal dispersion ($R_{56} := L/\gamma_0^2$) of all straight drift spaces $L$. This means, that given a characteristic relative energy spread of $\Delta_E := (E - E_0)/E_0$, ($E_0 = \gamma_0 mc^2$) we only transport longitudinal structures $l$ with

$$l \ll \frac{L\,\Delta_E}{\gamma_0^2} \tag{1}$$

over sections of length $L$.

The longitudinal PSD is $\Psi(\vec{z}) \equiv \Psi(q,p)$ with $\int_{\mathbb{R}^2} \Psi(\vec{z})d^2z = 1$, so that for every measure able set $A \subset \mathbb{R}^2$ the probability that any of the $N$ particles in the beam is in $A$ is $\int_A \Psi(\vec{z})d^2z = \mathcal{P}(A)$. The longitudinal configuration space density is $\rho(q) := \int_{\mathbb{R}} \Psi(q,p)dp$ so that the longitudinal charge density for a bunch of $N$ electrons is $-eN\rho$. The ultra-relativistic limit implies that $\rho(q)$ is frozen outside the magnetic chicanes.

Our models includes a self-consistent (in the mean field approximation) LSC with charge density as the source term in the Poisson equation. Therefore the map $\vec{M} : \vec{z}_i \mapsto \vec{z}_f = \vec{M}(\vec{z}_i)$ through the single bunch compression stage depends on $\rho$ and thus on $\Psi$. We denote the functional dependence by $[\cdot]$, so we write $\vec{M}[\Psi] : \vec{z}_i \mapsto \vec{z}_f = \vec{M}[\Psi](\vec{z}_i)$. All maps in this model are symplectic automorphisms of phase space and i.p., measure preserving. This implies that within the model the evolution of the PSD is a time-discrete Vlasov evolution:

$$\Psi_{\mathrm{f}} = \Psi_{\mathrm{i}} \circ \vec{M}[\Psi_{\mathrm{i}}]^{-1} \tag{2}$$

Following the discussion above and in Section 1 we formulate the basic model as *single bunch compressor stage* := { (long) *non-dispersive LinAcc + LSC* } followed by { (short) *magnetic chicane* }, i.e.

$$\vec{M}[\Psi] = \vec{D}^{\mathrm{drift}}_{\text{magnetic chicane}} \circ \vec{K}^{\mathrm{kick}}_{\text{cavities+LSC}}[\Psi]\ . \tag{3}$$

The structure of the maps $\vec{D}$ and $\vec{K}$ is discussed in greater detail in the following two subsections.

In the case of several single bunch compression stages, we iterate $\Psi_m = \Psi_{m-1} \circ \vec{M}_m[\Psi_{m-1}]^{-1}$.

The same base models is discussed in the contribution[6] by Ph. Amstutz.

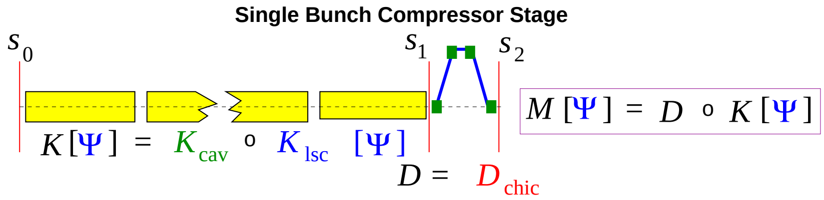


Fig. 1. The base model.

## 2.1. *The (long) LinAcc/LSC Part* $K[\Psi]$

As mentioned before we model this as a non-dispersive straight section so that $\rho(q)$ frozen. Then the transfer map is a *kick map* with a non-collective (RF-cavity) and Poisson-type collective (LSC) component.

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q \\ p + k_{\mathrm{cav}}(q) + k_{\mathrm{poi}}[\Psi](q) \end{pmatrix}\ , \tag{4}$$

$$\Psi_{\mathrm{f}}(q,p) = \Psi_{\mathrm{i}}(q, p - k_{\mathrm{cav}}(q) - k_{\mathrm{poi}}[\Psi_{\mathrm{i}}](q)) \ . \tag{5}$$

All kicks commute therefore the actual distribution of the RF cavities inside the linac does not matter within the boundaries of our model. Moreover, since kicks change only $p$ (modify only the $p$-density) and since the Poisson-type kick function $k_{\mathrm{poi}}$ depends only on $\Psi$ via the $q$-density $\rho$ ($p$-density integrated out), the LSC kick only depends on the initial $\Psi_{\mathrm{i}}$ of the linac section under consideration. This is explained in greater detail in[6]. Moreover, although the transverse dimensions vary (betatron envelope) over the linac, the commutativity of kicks grants that an averaged (along the linac beamline) Greens function $\langle G_{\mathrm{poi}} \rangle$ exists, so that

$$k_{\mathrm{poi}}[\Psi_{\mathrm{i}}](q) := L_k \int_{\mathbb{R}^2} \langle G_{\mathrm{poi}} \rangle (q,q') \Psi_{\mathrm{i}}(q',p') \, dq' dp' \equiv L_k \int_{\mathbb{R}} \langle G_{\mathrm{poi}} \rangle (q,q') \rho_{\mathrm{i}}(q') \, dq' \ , \tag{6}$$

where $L_k$ is the length of the linac. For convenience we introduce the linear operator $\mathfrak{K}_{\mathrm{poi}}$ that maps the PSDs into the kick-functions by

$$(\mathfrak{K}_{\mathrm{poi}} \, \Psi)(q) := k_{\mathrm{poi}}[\Psi](q) \ . \tag{7}$$

We assume an axially symmetric transverse PSD. Several Greens functions for various axially symmetric transverse shapes (uniform disk, delta-function, round Gaussian) are discussed in[6]. The integral cavity kick function is given by $k_{\mathrm{cav}}(q) = \tilde{h}q + O(q^2)$, where $\tilde{h} = hE_{\mathrm{ref}}$ is the absolute rf-chirp due to the linac given the relative (more familiar) rf-chirp $h$ which makes sense, however, only for zero acceleration ($E_{\mathrm{ref}} \equiv E_{\mathrm{i}} \equiv E_{\mathrm{f}}$). If $eV$ is the energy gain due to the vector sum amplitude of the cavities in the linac, $q_\phi$ is the distance of the the reference particle to the (vector sum) on-crest particle, and $\lambda_{\mathrm{rf}}$ is the (common) wavelength of the accelerating mode, then $\tilde{h} = -2\pi eV/\lambda_{\mathrm{rf}} \sin(2\pi/\lambda_{\mathrm{rf}} q_\phi)$.

The collective component of this kick map can potentially introduce an energy modulation due to an initial modulation of the spatial density.

### 2.2. *The (short) Magnetic Chicane Part D*

We assume that the chicane is short compared to the linac. We therefore neglect LSC inside the chicane. Furthermore we neglect here all synchrotron radiation effects, i.p. CSR. Then the transfer map through the chicane is a (generalized[‡]) *drift map*:

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q + \lambda(p) \\ p \end{pmatrix} \ . \tag{8}$$

It follows that the $p$-density is frozen inside the chicane within the limits of our model.

$$\Psi_{\mathrm{f}}(q,p) = \Psi_{\mathrm{i}}(q - \lambda(p), p) \ . \tag{9}$$

[‡]not a physical free space drift, but a map that only changes $q$ as a function of $p$

Here

$$\lambda(p) := L_d\, p + O(p^2)\ ,\ L_d := \frac{R_{56}}{E_{\text{ref}}}\ , \tag{10}$$

where $E_{\text{ref}}$ is the reference energy position of the chicane (downstream of our accelerating linac $E_{\text{ref}} = E_{\text{f}}$). With an initially unchirped beam, an upstream linac with absolute rf-chirp $\tilde{h} < 0$, final energy $E_{\text{f}}$, the (relative) beam chirp at the entrance of the chicane is $h := \tilde{h}/E_{\text{f}}$ and the beam is compressed by a factor

$$C := (1 + hR_{56})^{-1} = (1 + \tilde{h}L_d)^{-1}\ . \tag{11}$$

The drift map through the magnetic chicane can potentially introduce a modulation of the spatial density due to an initial energy modulation. Thus the interplay of LSC and the dispersive chicane modifies an initial spatial density modulation. The amplitudes of some Fourier components may be amplified while others are damped. As we will see, at higher order potentially harmonics of the incoming oscillations are generated. Micro-bunching is the amplification of small initial modulations (and/or the generation of new harmonics) through successive bunch compressor stages.

## 3. A suitable Perturbation Theory for Time-Discrete Vlasov Systems

We now want to establish the mathematical framework for setting up a suitable perturbation theory for a time-discrete Vlasov evolution. Although this is beyond the scope of this paper, we finally want to arrive at a theory that allows for specifying rigorous error bounds. Therefore it is probably helpful to chose a proper mathematical framework right from the beginning.

### 3.1. *The Perron Frobenius Operator*

Let $\vec{M} : \mathbb{R}^2 \to \mathbb{R}^2$ be a measure preserving and invertible (here in fact: symplectic), and sufficiently smooth map with smooth inverse. In other words, $\vec{M}$ is a symplectic diffeomorphism of class $\mathcal{C}^m$, $\vec{M} \in \mathbf{Symp}(\mathbb{R}^2) \cap \mathcal{C}^m(\mathbb{R}^2)$, for some $m > 0$. Moreover, let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ in some Banach space of absolute Lebesgue integrable functions $\mathcal{W}(\mathbb{R}^2, \mathbb{R})$. At this stage $\mathcal{W}$ could be just $\mathcal{L}^1$ but later we might need some (weak) smoothness, e.g. $\Psi \in \mathcal{W}^1_m$ for some $m > 0$, where $\mathcal{W}^1_m \subset \mathcal{L}^1$ is the *Sobolev space* of $m$-times weakly differentiable absolute integrable functions. We will require a certain minimal $m$ to ensure that certain terms in our $n$-th order perturbation theory make sense (as it will turn out: $m = n$), but we might still require larger $m > n$ for error bounds later! We want to mention here that this model is not very likely to ab applicable to strongly "curled up" reference bunches as they sometimes appear in FELs. Thus numerics capable of describing exotic FEL type PSDs are needed. See Ph. Amstutz's contribution[6]!

Since $\vec{M}$ is measure preserving, and chosen sufficiently smooth, the composition $\Psi \circ \vec{M}^{-1}$ is also in $\mathcal{W}(\mathbb{R}^2, \mathbb{R})$, and thus we may define for every such $\vec{M}$ the corresponding *Perron Frobenius Operator*[7,8]

$$\mathcal{M} : \mathcal{W}(\mathbb{R}^2, \mathbb{R}) \to \mathcal{W}(\mathbb{R}^2, \mathbb{R}) \ , \Psi \mapsto \mathcal{M}\Psi := \Psi \circ \vec{M}^{-1} \ . \tag{12}$$

One can easily convince oneself that $\mathcal{M}$ is a linear operator:

$$\mathcal{M}\left(\mu\Psi + \nu\Phi\right) = \mu\mathcal{M}\Psi + \nu\mathcal{M}\Phi \ , \ \ \forall \ \ \Psi, \Phi \in \mathcal{W} \ \ \& \ \ \mu, \nu \in \mathbb{R} \ . \tag{13}$$

Here have introduced the abbreviation $\mathcal{W}$ meaning $\mathcal{W}(\mathbb{R}^2, \mathbb{R})$. Now we write $\mathcal{M} \in \mathrm{lin}(\mathcal{W}, \mathcal{W})$ to denote that $\mathcal{M}$ belongs to the set of linear operators from $\mathcal{W}$ to $\mathcal{W}$.

$\mathcal{M}\Psi$ describes the Liouville evolution of $\Psi$ through $\vec{M}$.

$\mathcal{M}[\Psi]\Psi$ describes the (self-consistent) Vlasov evolution of $\Psi$ through $\vec{M}[\Psi]$ which in turn depends (in a functional way) on $\Psi$.

We want to make a point here that $\mathcal{M}[\Psi]$ is not linear in $\Psi$. However, if $\vec{M}[\Psi]$ is sufficiently regular in its functional dependence on $\Psi$, we might hope to be able to linearize $\mathcal{M}[\cdot]$ around any given $\Psi_0$.

### 3.2. *The Total (Frechet) Derivative*

Let $\|\cdot\|_{\mathrm{op}}$ be a suitable operator norm on $\mathrm{lin}(\mathcal{W}, \mathcal{W})$. If a linear operator $\mathcal{M}'[\Psi_0] \in \mathrm{lin}(\mathcal{W}, \mathrm{lin}(\mathcal{W}, \mathcal{W}))$ exists, so that $\mathcal{M}[\Psi_0 + \phi] = \mathcal{M}[\Psi_0] + \mathcal{M}'[\Psi_0] \cdot \phi + o(\phi)$, in other words:

$$\lim_{\phi \to 0} \frac{\|\mathcal{M}[\Psi_0 + \phi] - \mathcal{M}[\Psi_0] - \mathcal{M}'[\Psi_0] \cdot \phi\|_{\mathrm{op}}}{\|\phi\|_{\mathcal{W}}} = 0 \ ,$$

then $\mathcal{M}'[\Psi_0]$ is called the generalized total (Frechet) derivative[9] of $\mathcal{M}[\cdot] : \mathcal{W} \to \mathrm{lin}(\mathcal{W}, \mathcal{W})$, $\Psi \mapsto \mathcal{M}[\Psi]$ at $\Psi_0$.

Here and in the following we use a $\cdot$ to bind the function $\phi$ to the linear operator $\mathcal{M}'[\Psi_0]$ which yields another linear operator (a new Perron-Frobenius operator) to be applied to other functions without the $\cdot$. We hope that this makes expressions like $(\mathcal{M}'[\Psi_0] \cdot \phi)\Phi_0$ (and similar) appear less obscure!

Higher order Frechet derivatives can be defined in an analog way, e.g. $\mathcal{M}''[\Psi_0] \in \mathrm{lin}(\mathcal{W}, \mathrm{lin}(\mathcal{W}, \mathrm{lin}(\mathcal{W}, \mathcal{W})))$, with $\mathcal{M}'[\Psi_0 + \phi] = \mathcal{M}'[\Psi_0] + \mathcal{M}''[\Psi_0] \cdot \phi + o(\phi)$, etc.

## 4. The Linearized Vlasov Evolution for One BC Stage

We start with a bunch compression stage with transfer map

$$\vec{M}[\Psi] = \vec{D} \circ \vec{K}_{\mathrm{cav}} \circ \vec{K}_{\mathrm{poi}}[\Psi] =: \vec{L} \circ \vec{K}_{\mathrm{poi}}[\Psi] \ , \tag{14}$$

where $\vec{L} := \vec{D} \circ \vec{K}_{\mathrm{cav}}$ combines the non-collective (lattice) part of the transfer map, and a weakly smooth *reference* bunch PSD $\Psi_0 \in \mathcal{W}_1^1$, mapped by the BC stage to

$$\Psi_1 := \mathcal{M}[\Psi_0]\Psi_0 \ . \tag{15}$$

Then the linearized Vlasov Evolution of a perturbation $\phi_0 \in \mathcal{W}_1^1$ along with the unperturbed evolution $\Psi_0 \to \Psi_1$ reads

$$\begin{aligned}
&\mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,(\Psi_0 + \varepsilon\phi_0) \\
&= \mathcal{M}[\Psi_0]\,\Psi_0 + \varepsilon\left(\mathcal{M}[\Psi_0]\,\phi_0 + (\mathcal{M}'[\Psi_0]\cdot\phi_0)\;\Psi_0\right) + O(\varepsilon^2) \qquad (16)\\
&=: \Psi_1 + \varepsilon\phi_1 + O(\varepsilon^2)\;. \qquad (17)
\end{aligned}$$

We note that because of the linearity of the Poisson equation, $\vec{K}_{\rm poi}[\Psi + \Phi] = \vec{K}_{\rm poi}[\Psi] \circ \vec{K}_{\rm poi}[\Phi] = \vec{K}_{\rm poi}[\Phi] \circ \vec{K}_{\rm poi}[\Psi]$ and also $\vec{K}_{\rm poi}[\Psi + \Phi]^{-1} = \vec{K}_{\rm poi}[\Psi]^{-1} \circ \vec{K}_{\rm poi}[\Phi]^{-1} = \vec{K}_{\rm poi}[\Phi]^{-1} \circ \vec{K}_{\rm poi}[\Psi]^{-1}$. Before we proceed we introduce the following abbreviations:

$$Q(\vec{z} \equiv (q,p)) := q \equiv (\vec{z})_1\;, \qquad (18)$$

$$Q_L(\vec{z}) := Q(\vec{L}^{-1}(\vec{z})) \equiv (\vec{L}^{-1}(\vec{z}))_1 \;\;\&\;\; P_L(\vec{z}) := (\vec{L}^{-1}(\vec{z}))_2\;. \qquad (19)$$

The $O(\varepsilon)$ term of Eq. (16) has two contributions. The first is $\mathcal{M}[\Psi_0]\phi_0$. It arises because of the linearity of $\mathcal{M}[\Psi_0]$ and has no perturbative nature. The second contains the Frechet derivative of $\mathcal{M}[\cdot]$ at $\Psi_0$ and can be computed via

$$\begin{aligned}
&\mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,\Psi_0 \\
&= \Psi_0 \circ \vec{K}_{\rm poi}[\Psi_0 + \varepsilon\phi_0]^{-1} \circ \vec{L}^{-1} \\
&= \Psi_0\big(Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L) - \varepsilon k_{\rm poi}[\phi_0](Q_L)\big) \qquad (20)\\
&= \Psi_0\big(Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L)\big) \\
&\quad - \varepsilon\partial_p\Psi_0\big|_{Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L)} \cdot k_{\rm poi}[\phi_0](Q_L) + O(\varepsilon^2)\;. \qquad (21)
\end{aligned}$$

We note that up to Eq. (20) no approximations have been made at all. Equation (21) is the first order Taylor polynomial of $\Psi_0$ w.r.t. $p$ around $\vec{M}[\Psi_0]^{-1}(\vec{z})$. The upper term in Eq. (21) is just $\Psi_1 = \mathcal{M}[\Psi_0]\Psi_0$. The $O(\epsilon)$ part of the lower term is $(\mathcal{M}'[\Psi_0]\cdot\phi_0)\;\Psi_0$, i.e. the action of the linearized PF operator on $\Psi_0$. Now finally the linearly evolved perturbation $\phi_1$ is given by

$$\begin{aligned}
\phi_1(\vec{z}) &= \phi_0\big(Q_L(\vec{z}), P_L(\vec{z}) - k_{\rm poi}[\Psi_0](Q_L(\vec{z}))\big) \\
&\quad - \partial_2\Psi_0\big(Q_L(\vec{z}), P_L(\vec{z}) - k_{\rm poi}[\Psi_0](Q_L(\vec{z}))\big) \cdot k_{\rm poi}[\phi_0](Q_L(\vec{z}))\;, \qquad (22)
\end{aligned}$$

where and $\partial_2$ means partial derivative w.r.t. the 2nd component $p$. This can be rephrased in slightly more general form

$$\phi_1 = \phi_0 \circ \vec{M}[\Psi_0]^{-1} - \partial_2\Psi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot k_{\rm poi}[\phi_0] \circ Q \circ \vec{L}^{-1}\;. \qquad (23)$$

One easily identifies

$$(\mathcal{M}'[\Psi_0]\cdot f) : g \mapsto -\partial_2 g \circ \vec{M}[\Psi_0]^{-1} \cdot (\mathfrak{K}_{\rm poi} f) \circ Q \circ \vec{L}^{-1} \qquad (24)$$

with $f, g \in \mathcal{W}_1^1$ is the linearization of $\mathcal{M}[\cdot]$ around $\Psi_0$ at some deviation $f$. The actual Frechet derivative $\mathcal{M}'[\Psi_0]$ is the linear function that maps every $f$ into $(\mathcal{M}'[\Psi_0]\cdot f)$. We note that no explicit knowledge of the unperturbed $\Psi_1$ is needed for the single stage linear Vlasov evolution!

If one restricts $\vec{D}$ and $\vec{K}_{\rm cav}$ to linear functions, so that $\vec{L}(\vec{z}) = \underline{L}\vec{z}$, with

$$\underline{L} \equiv \underline{D}\,\underline{K}_{\rm cav} := \begin{pmatrix} C^{-1} & L_d \\ \tilde{h} & 1 \end{pmatrix} , \tag{25}$$

$$Q_L^1(\vec{z}) \equiv \underline{Q}_L\vec{z} := (\underline{L}^{-1}\vec{z})_1 = q - L_d p , \tag{26}$$

$$P_L^1(\vec{z}) \equiv \underline{P}_L\vec{z} := (\underline{L}^{-1}\vec{z})_2 = -\tilde{h}q + p/C , \tag{27}$$

then one finds for a model with linear "RF-cavities" and a chicane with purely linear longitudinal dispersion

$$\begin{aligned} \phi_1(\vec{z}) &= \phi_0\big(\underline{Q}_L\vec{z}, \underline{P}_L\vec{z} - k_{\rm poi}[\Psi_0](\underline{Q}_L\vec{z})\big) \\ &\quad - \partial_2\Psi_0\big(\underline{Q}_L\vec{z}, \underline{P}_L\vec{z} - k_{\rm poi}[\Psi_0](\underline{Q}_L\vec{z})\big) \cdot k_{\rm poi}[\phi_0](\underline{Q}_L\vec{z}) . \end{aligned} \tag{28}$$

## 5. *n*th Order Pert. Expansion

Now let $\Psi_0, \phi_0 \in \mathcal{W}_n^1$. Extending Eq. (20) to $n$th order yields

$$\begin{aligned} &\mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,\Psi_0 \\ &= \Psi_0\big(Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L)\big) \\ &\quad \sum_{k=1}^{n} \frac{(-\varepsilon)^k}{k!}\partial_p^k\Psi_0\big|_{Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L)} \cdot \big(k_{\rm poi}[\phi_0](Q_L)\big)^k + O(\varepsilon^{n+1}) . \end{aligned} \tag{29}$$

At each order $> 0$ we obtain an additional term from expanding

$$\begin{aligned} &\mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,\varepsilon\phi_0 \\ &= \varepsilon\phi_0\big(Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L)\big) \\ &\quad - \sum_{k=1}^{n-1} \frac{(-\varepsilon)^{k+1}}{k!}\partial_p^k\phi_0\big|_{Q_L, P_L - k_{\rm poi}[\Psi_0](Q_L)} \cdot \big(k_{\rm poi}[\phi_0](Q_L)\big)^k + O(\varepsilon^{n+1}) . \end{aligned} \tag{30}$$

Thus we obtain at order $n$:

$$\mathcal{M}[\Psi_0 + \varepsilon\phi_0]\,(\Psi_0 + \varepsilon\phi_0) = \Psi_1 + \sum_{k=1}^{n} \varepsilon^k\phi_{1,k} + O(\varepsilon^{n+1}) \tag{31}$$

$$\begin{aligned} \phi_{1,k} &:= \frac{(-1)^k}{k!}\partial_2^k\Psi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot (k_{\rm poi}[\phi_0] \circ Q_L)^k \\ &\quad + \frac{(-1)^{k-1}}{(k-1)!}\partial_2^{k-1}\phi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot (k_{\rm poi}[\phi_0] \circ Q_L)^{k-1} . \end{aligned} \tag{32}$$

In particular we have $\phi_{1,1} \equiv \phi_1$ (as in Sec. 4), and

$$\begin{aligned} \phi_{1,2}(\vec{z}) &= \frac{1}{2}\partial_2^2\Psi_0\big(Q_L(\vec{z}), P_L(\vec{z}) - k_{\rm poi}[\Psi_0](Q_L(\vec{z}))\big) \cdot \big(k_{\rm poi}[\phi_0](Q_L(\vec{z}))\big)^2 \\ &\quad - \partial_2^1\phi_0\big(Q_L(\vec{z}), P_L(\vec{z}) - k_{\rm poi}[\Psi_0](Q_L(\vec{z}))\big) \cdot \big(k_{\rm poi}[\phi_0](Q_L(\vec{z}))\big)^1 . \end{aligned} \tag{33}$$

Note that the lowest order "self modulation" term of $\phi_0$, namely $\partial_2\phi_0 \circ \vec{M}[\Psi_0]^{-1} \cdot k_{\rm poi}[\phi_0] \circ Q_L$ only enters at 2nd order! We note that no explicit knowledge of the unperturbed $\Psi_1$ is needed for the single stage $n$-th order Vlasov evolution!

## 6. Cascade of several BC Stages

We want to describe several stages truncated at order $n$. The first stage $\mathcal{M}_1$ maps $\Psi_0 + \varepsilon\phi_{0,1}$ into $\Psi_1 + \sum_{i=1}^n \varepsilon^i \phi_{1,i} + O(\varepsilon^{n+1})$. The second stage $\mathcal{M}_2$ already gets $\Psi_1 + \sum_{i=1}^n \varepsilon^i \phi_{1,i}$ as input and maps it into $\Psi_2 + \sum_{i=1}^n \varepsilon^i \phi_{2,i} + O(\varepsilon^{n+1})$. More generally: the $m+1$-st stage $\mathcal{M}_{m+1}$ maps $\Psi_m + \sum_{i=1}^n \varepsilon^i \phi_{m,i}$ into $\Psi_{m+1} + \sum_{i=1}^n \varepsilon^i \phi_{m+1,i} + O(\varepsilon^{n+1})$.

For convenience we will omit the stage-index in the map-like entities ($\mathcal{M}$, $\vec{L}$, $\vec{K}_{\rm poi}$, $Q_L$, $P_L$, etc.), and write $\sum_{i=0}^n \psi_{m,i}$ for $\Psi_m + \sum_{i=1}^n \phi_{m,i}$. The the $m+1$-st stage evolves

$$
\begin{aligned}
&\mathcal{M}\left[\sum_{i=0}^n \varepsilon^i \psi_{m,i}\right]\left(\sum_{l=0}^n \varepsilon^l \psi_{m,l}\right) \\
&= \sum_{i=0}^n \varepsilon^i \psi_{m,i}\left(Q_L, P_L - \sum_{l=0}^n \varepsilon^l k_{\rm poi}[\psi_{m,l}](Q_L)\right) \qquad (34)\\
&= \sum_{i=0}^n \varepsilon^i \psi_{m,i}\left(Q_L, P_L - k_{\rm poi}[\psi_{m,0}](Q_L) - \sum_{l=1}^n \varepsilon^l k_{\rm poi}[\psi_{m,l}](Q_L)\right) \\
&= \sum_{i=0}^n \varepsilon^i \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \partial_2^j \psi_{m,i}\big(Q_L, P_L - k_{\rm poi}[\psi_{m,0}](Q_L)\big) \cdot \\
&\qquad \cdot \left(\sum_{l=1}^{(n-i)//j} \varepsilon^l k_{\rm poi}[\psi_{m,l}](Q_L)\right)^j + O(\varepsilon^{n+1}) \qquad (35)\\
&= \sum_{i=0}^n \varepsilon^i \sum_{j=0}^{n-i} \frac{(-1)^j}{j!} \partial_2^j \psi_{m,i}\big(\vec{M}[\psi_{m,0}]^{-1}\big) \cdot \left(\sum_{l=1}^{(n-i)//j} \varepsilon^l k_{\rm poi}[\psi_{m,l}](Q_L)\right)^j + O(\varepsilon^{n+1}) . \qquad (36)
\end{aligned}
$$

Eq. (34) is again exact for truncated input and Eqs. (35) and (36) allow iteration along a cascade of several bunch compressor stages. In the latter two equations the upper sum-limit $(n-i)//j$ is defined as $(n-i)/j$ iff $j \neq 0$ and as 0 otherwise. Note that $j = 0$ implies that the sum over $l$ is taken to the zeroth power. For Example with $n = 2$ and $m = 2$ we find $\psi_{2,0} = \mathcal{M}_1[\psi_{1,0}]\psi_{1,0} = \mathcal{M}_1[\mathcal{M}_0[\psi_{0,0}]\psi_{0,0}]\ \mathcal{M}_0[\psi_{0,0}]\ \psi_{0,0}$, and

$$
\begin{aligned}
\psi_{2,1} = {} & \psi_{1,1}\big(Q_{L1}, P_{L1} - k_1[\psi_{1,0}](Q_{L1})\big) \\
& - \partial_2 \psi_{1,0}\big(Q_{L1}, P_{L1} - k_1[\psi_{1,0}](Q_{L1})\big) \cdot k_1[\psi_{1,1}](Q_{L1}) \,, \qquad (37)
\end{aligned}
$$

where $\psi_{1,0} = \mathcal{M}_0[\psi_{0,0}]\ \psi_{0,0}$ and

$$
\begin{aligned}
\psi_{1,1} = {} & \psi_{0,1}\big(Q_{L0}, P_{L0} - k_0[\psi_{0,0}](Q_{L0})\big) \\
& - \partial_2 \psi_{0,0}\big(Q_{L0}, P_{L0} - k_0[\psi_{0,0}](Q_{L0})\big) \cdot k_0[\psi_{0,1}](Q_{L0}) \,, \qquad (38)
\end{aligned}
$$

(analogously to Eq. (22)), and

$$\begin{aligned}\psi_{2,2} &= \psi_{1,2}\big(Q_{L1}, P_{L1} - k_1[\psi_{1,0}](Q_{L1})\big) \\ &\quad - \partial_2\psi_{1,1}\big(Q_{L1}, P_{L1} - k_1[\psi_{1,0}](Q_{L1})\big) \cdot k_1[\psi_{1,1}](Q_{L1}) \\ &\quad + \frac{1}{2}\partial_2^2\psi_{1,0}\big(Q_{L1}, P_{L1} - k_1[\psi_{1,0}](Q_{L1})\big) \cdot k_1[\psi_{1,1}](Q_{L1}) \\ &\quad - \partial_2\psi_{1,1}\big(Q_{L1}, P_{L1} - k_1[\psi_{1,0}](Q_{L1})\big) \cdot k_1[\psi_{1,2}](Q_{L1})\,, \end{aligned} \tag{39}$$

where $\psi_{1,2}$ is given in analogy to Eq. (33) by

$$\begin{aligned}\psi_{1,2} &= \frac{1}{2}\partial_2^2\psi_{0,0}\big(Q_{L0}, P_{L0} - k_0[\psi_{0,0}](Q_L)\big) \cdot \big(k_0[\psi_{0,1}](Q_{L0})\big)^2 \\ &\quad - \partial_2^1\psi_{0,1}\big(Q_{L0}, P_{L0} - k_0[\psi_{0,0}](Q_L)\big) \cdot \big(k_0[\psi_{0,1}](Q_{L0})\big)\,. \end{aligned} \tag{40}$$

We note that no explicit knowledge of the unperturbed $\Psi_m + 1$ is needed for the multi-stage $n$-th order Vlasov evolution!

## 7. Gain Functions

To study the properties of and the potential damage due to micro-bunching in multi-stage bunch compression lattices, so called gain functions have been introduced. They are typically based on the quotients of power spectra taken at compression-corrected wavelengths. We will pursue here a similar though slightly more general concept: Starting from a cascade of $m$ bunch compressors $\vec{M}_1,\ldots, \vec{M}_m$, a given unperturbed PSD $\Psi_0$, all sufficiently regular so that the sequence $\Psi_0 \mapsto \ldots \mapsto \Psi_m$ can all be computed, and an initial $O(\varepsilon)$ perturbation $\phi_0 \equiv \psi_{0,1}$ with projected spatial density $\rho_{0,1}(q)$ and Fourier transform $\hat{\rho}_{0,1}(\kappa)$, one may define the most general $n$-th order $m$-stage (amplitude-) gain function:

$$g^{(n,m)}[\Psi_0, \vec{M}_0, \cdots, \vec{M}_m; \phi_0](\kappa_{\mathrm{i}}, \kappa_{\mathrm{f}}) := \frac{\hat{\rho}_{m,n}(\kappa_{\mathrm{f}})}{\hat{\rho}_{0,1}(\kappa_{\mathrm{i}})}\,. \tag{41}$$

In addition one may define the accumulated (over all orders $\le n$) $m$-stage (amplitude-) gain function:

$$\Gamma^{(n,m)}[\Psi_0, \vec{M}_0, \cdots, \vec{M}_m; \phi_0](\kappa_{\mathrm{i}}, \kappa_{\mathrm{f}}) := \sum_{k=1}^{n} g^{(k,m)}[\ldots](\kappa_{\mathrm{i}}, \kappa_{\mathrm{f}})\,. \tag{42}$$

If the chirp of the reference PSDs $(\Psi_0, \ldots, \Psi_{m-1})$ does only vary weakly over the bunchlength, and if generation of harmonics is neglected, it is often convenient to define the compression corrected, absolute (power spectral) gain:

$$\tilde{g}^{(n,m)}[\Psi_0, \vec{M}_0, \cdots, \vec{M}_m; \phi_0](\kappa) := \big|g^{(n,m)}[\ldots](\kappa, \kappa\cdot(C_0\cdots C_m))\big|^2 \tag{43}$$

$$\tilde{\Gamma}^{(n,m)}[\Psi_0, \vec{M}_0, \cdots, \vec{M}_m; \phi_0](\kappa) := \big|\Gamma^{(n,m)}[\ldots](\kappa, \kappa\cdot(C_0\cdots C_m))\big|^2\,. \tag{44}$$

## 8. A First Example: Infinitely Long Bunch

In this section we will very briefly touch the example of an "infinitely" long bunch. We define $\xi_{\mu,\sigma}(x) := (\sigma\sqrt{2\pi})^{-1}\exp(-(x-\mu)^2/2/\sigma^2)$ as a Gaussian with expectation $\mu$ and variance $\sigma^2$, and $\Lambda_t(x) := 1/(2t+\delta)$, $\delta \ll 2t$ over the interval $[-t,t]$ and $\Lambda_t(x) = 0$ for $|x| \gg t$. We assume $\Lambda$ is sufficiently smooth, but with "infinitely long" we mean that we ignore all edge effects. Our initial unperturbed PSD is

$$\Psi_0(q,p) \equiv \psi_{0,0}(q,p) = \Lambda_{t_0}(q)\xi_{\mu_0(q),\sigma_0}(p) \ . \tag{45}$$

We start at the reference energy $E_0$ and with no initial chirp $d\mu/dq$, so that $\mu_0(q) \equiv 0$. We assume $\sigma_0 = \text{const.} > 0$ and $\sigma_0/E_0 \ll (\kappa t_0)^{-1}$ for all wavenumbers $\kappa$ under study. Moreover we assume that the bunch has a characteristic transverse size $a \ll t_0$. Then $\rho_{0,0}(q) = \Lambda_{t_0}(q)$ and furthermore $k_{\text{poi}}[\Psi_0] \approx 0$ deep enough *inside* the bunch and for any decent $\langle G_{\text{poi}}\rangle$. We set $\vec{D}$, $\vec{K}_{\text{cav}}$ linear, $L_d > 0$, $\tilde{h} < 0$, so that $\underline{L}$ describes a linear bunch compressor with compression $C > 1$. Then $\Psi_1(\vec{z}) \equiv \psi_{1,0}(\vec{z}) = \Lambda_{t_1}(q)\xi_{\mu_1(q),\sigma_1}(p) \approx \Psi_0(\underline{Q}_L\vec{z}, \underline{P}_L\vec{z})$, i.p. $\mu_1(q) = \tilde{h}q$, $t_1 \approx t_0/C$.

Now set the initial perturbation

$$\phi_0(q,p) \equiv \psi_{0,1}(q,p) = \eta\psi_{0,0}(q,p)\cos(\kappa_{\text{i}}q) \ , \tag{46}$$

with $0 \leq \eta < 1$, so that $\Psi_0 + \phi_0 \geq 0$ and $\int_{\mathbb{R}^2}(\Psi_0+\phi_0)\,dqdp \approx 1$ for $\kappa_{\text{i}}t_0 \gg 1$. With this perturbation,

$$\rho_{0,1}(q) = \eta\Lambda_{t_0}(q)\cos(\kappa_{\text{i}}q) \ . \tag{47}$$

In order to proceed we need a model for the LSC impedance[6]. Here we choose the mean force on a charged $a$-disk due to a charged $a$-disk: $\hat{G}_{\text{poi}}(\kappa) \propto \frac{ia^2}{\kappa}\big(1 - 2I_1(|\kappa|a)K_1(|\kappa|a)\big)$, but more ways to model the influence of the transverse beam shape exist[6]. Now we can compute the kick function for values of $|q| \ll t_0$, i.e. deep inside the bunch:

$$k_{\text{poi}}[\phi_0](q) = \frac{\eta}{2t_0}\Im\hat{G}_{\text{poi}}(\kappa_{\text{i}})\sin(\kappa_{\text{i}}q) \ . \tag{48}$$

If we now choose $n = 2$ (and $m = 1$), and define $\hat{G}_{\text{i}} := \Im\hat{G}_{\text{poi}}(\kappa_{\text{i}})$, we find for our single stage bunch compressor

$$\begin{aligned}\psi_{1,1} &\approx -\partial_2\psi_{0,0}(Q_L^1, P_L^1)\cdot\frac{\eta}{2t_0}\hat{G}_{\text{i}}\sin(\kappa_{\text{i}}Q_L^1)\\ &\quad + \psi_{0,1}(Q_L^1, P_L^1) \qquad (49)\\ \psi_{1,2} &\approx \frac{1}{2}\partial_2^2\psi_{0,0}(Q_L^1, P_L^1)\cdot\big(\frac{\eta}{2t_0}\hat{G}_{\text{i}}\sin(\kappa_{\text{i}}Q_L^1)\big)^2\\ &\quad - \partial_2\psi_{0,1}(Q_L^1, P_L^1)\cdot\frac{\eta}{2t_0}\hat{G}_{\text{i}}\sin(\kappa_{\text{i}}Q_L^1) \qquad (50)\end{aligned}$$

Note that since $\xi'_{\mu,\sigma}(p) = -\frac{p-\mu}{\sigma^2}\xi_{\mu,\sigma}(p)$ and $\xi''_{\mu,\sigma}(p) = \big(\frac{(p-\mu)^2}{\sigma^4} - \frac{1}{\sigma^2}\big)\xi_{\mu,\sigma}(p)$, all functions above can be evaluated explicitly.

Now let the chirp $\tilde{h}$ be so weak, that the compression $C = 1/(1+\tilde{h}L_d) \approx 1$, despite finite $L_d$. Then $P_L^1(q,p) \approx p$, and since $\Lambda_{t_0}(q)$ is constant well inside the bunch, the projections $\psi \circ \vec{L} \to \rho$ can be performed

$$\begin{aligned}\rho_{1,1}(q) \approx & \frac{\eta}{t_0}\cos(\kappa_{\mathrm{i}} q)\int_0^\infty \xi_{0,\sigma}(p)\cos(\kappa_{\mathrm{i}} L_d p)\,dp \\ & - \frac{\eta\hat{G}_{\mathrm{i}}}{2t_0^2}\cos(\kappa_{\mathrm{i}} q)\int_0^\infty \frac{p}{\sigma^2}\,\xi_{0,\sigma}(p)\sin(\kappa_{\mathrm{i}} L_d p)\,dp\end{aligned} \tag{51}$$

since $\xi'_{0,\sigma}(p)$ is odd, and

$$\begin{aligned}\rho_{1,2}(q) \approx & \frac{\eta^2\hat{G}_{\mathrm{i}}^2}{16t_0^3}\cos(2\kappa_{\mathrm{i}} q)\int_0^\infty \left(\frac{p^2}{\sigma^4} - \frac{1}{\sigma^2}\right)\xi_{0,\sigma}(p)\cos(2\kappa_{\mathrm{i}} L_d p)\,dp \\ & - \frac{\eta^2\hat{G}_{\mathrm{i}}}{4t_0^2}\cos(2\kappa_{\mathrm{i}} q)\int_0^\infty \frac{p}{\sigma^2}\,\xi_{0,\sigma}(p)\sin(\kappa_{\mathrm{i}} L_d p)\,dp\end{aligned} \tag{52}$$

since $\xi''_{0,\sigma}(p)$ is even. The integral $\int_0^\infty \cos(2\kappa_{\mathrm{i}} L_d p)\xi_{0,\sigma}(p)$ can be found in[10], the other two can be derived from there using integration by parts. Thus we find

$$\rho_{1,1}(q) \approx \frac{\eta}{2t_0}\exp\left(-\frac{1}{2}\kappa_{\mathrm{i}}^2 L_d^2\sigma^2\right)\left(1 - \frac{\hat{G}_{\mathrm{i}}\kappa_{\mathrm{i}} L_d}{2t_0}\right)\cos(\kappa_{\mathrm{i}} q)\ , \tag{53}$$

and

$$\rho_{1,2}(q) \approx \frac{\eta^2\hat{G}_{\mathrm{i}}\kappa_{\mathrm{i}} L_d}{2t_0^2}\exp\left(-2\kappa_{\mathrm{i}}^2 L_d^2\sigma^2\right)\left(\frac{\hat{G}_{\mathrm{i}}\kappa_{\mathrm{i}} L_d}{2t_0} - 1\right)\cos(2\kappa_{\mathrm{i}} q)\ . \tag{54}$$

The gain functions are easily computed since the only $q$-dependence (neglecting the edge effects) is in the cos-terms of argument $\kappa_{\mathrm{i}} q$ and $2\kappa_{\mathrm{i}} q$ for first an second order contributions respectively. We want to mention here that already for a single stage bunch compressor and up to only 2nd order we drive a 2nd harmonic ($\kappa_{\mathrm{i}} \to 2\kappa_{\mathrm{i}}$). The usual compression corrected gain, even at 2nd order $\tilde{\Gamma}^{(2,1)}(\kappa)$ cannot capture this.

This is still work in progress and we apologize this incompletely worked out example. The original poster contained some bugs which we hope to have removed in this contribution. There is more interesting work to do and more publications to come!

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